数理解析研究所講究録 1343 巻 2003 年 31-45

DEFINABLE G-FIBER BUNDLES AND DEFINABLE C^rG-FIBER BUNDLES

TOMOHIRO KAWAKAMI
川上智博 (和歌山大学)

ABSTRACT. Let G be a compact definable group and $f, h : X \to Y$ definable G-maps between definable G-sets. We prove that if X is compact, η is a definable G-fiber bundle over Y and f and h are G-homotopic, then $f^*(\eta)$ and $h^*(\eta)$ are definably G-isomorphic.

Let G be a compact subgroup of $GL_n(\mathbb{R})$ and $f, h : X \to Y$ definable C^rG maps between definable C^rG manifolds. We show that if X is compact and affine, η is a definable C^rG-fiber bundle over Y and f and h are definably C^rG-homotopic, then $f^*(\eta)$ and $h^*(\eta)$ are definably C^rG-isomorphic.

1. INTRODUCTION

Let $\mathcal{M}$ denote an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. The term "definable" means "definable with parameters in $\mathcal{M}$". In this paper, we are concerned with homotopy property of definable G-fiber bundles and definable C^rG-fiber bundles when $1 \leq r < \infty$. General references on o-minimal structures are [6], [8], see also [18]. Further properties and constructions of them are studied in [7], [9], [17]. Every definable category is a generalization of the semialgebraic category and the definable category on $\mathcal{R}$ coincides the semialgebraic one.

A group G is a *definable group* if G is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable. A *definable G-set* means a G-invariant definable subset of some representation of G. We use a definable space as in the sense of [6], and every definable set is a definable space in this sense. Throughout this paper, definable maps between definable spaces are assumed to be continuous.

Theorem 1.1. *Let G be a compact definable group. Suppose that $\eta = (E, p, Y, F, K)$ is a definable G-fiber bundle over a definable G set Y and $f, h : X \to Y$ are definable G-maps between definable G-sets. If X is compact and f and h are G-homotopic, then $f^*(\eta)$ and $h^*(\eta)$ are definably G-isomorphic.*

Two definable G-maps $f, h : X \to Y$ between definable G-sets are *definably G-homotopic* if there exists a definable G-map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = h(x)$ for all $x \in X$, where the action on $[0, 1]$ is trivial. By 1.2 [11], two definable G-maps in Theorem 1.1 are definably G-homotopic.

2000 *Mathematics Subject Classification.* 14P10, 14P20, 57R22, 57R35, 57S10, 57S15, 58A05, 58A07, 03C64.

Keywords and Phrases. Definable G-sets, definable G-fiber bundles, definable G-vector bundles, o-minimal, compact definable groups, definable C^rG-manifolds, definable C^rG-fiber bundles, definable C^rG-vector bundles.

In the rest of this paper except section 2, G and K denote compact subgroups of $GL_n(\mathbb{R})$. It is known that they are compact algebraic subgroups of $GL_n(\mathbb{R})$ (e.g. 2.2 [16]).

Let Ω be a representation of G and $k \in \mathbb{N}$. Then we can consider the universal G-vector bundle $\gamma(\Omega, k)$ associated with Ω and k (see Definition 3.1). A definable G-vector bundle $\eta = (E, p, X)$ over a definable G-set X is called *strongly definable* if there exist a representation Ω of G and a definable G-map $f : X \to G(\Omega, k)$ such that η is definably G-isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η. The following result is a definable version of 1.1 [3].

Theorem 1.2. *Every definable G-vector bundle over a definable G-set is strongly definable.*

Let X be a definable G-set. Let $Vect^G_{def}(X)$ (respectively $Vect^G(X)$) denote the set of definable G-isomorphism (respectively G-isomorphism) classes of definable G-vector bundles (respectively G-vector bundles) over X. Then there is a canonical map $\kappa : Vect^G_{def}(X) \to Vect^G(X)$ which sends the definable G-isomorphism class $[\eta]^G_{def}$ of a definable G-vector bundle η over X to the G-isomorphism class $[\eta]^G$ of η.

Theorem 1.3. *Let X be a definable G-set. Then the map $\kappa : Vect^G_{def}(X) \to Vect^G(X)$ defined by $\kappa([\eta]^G_{def}) = [\eta]^G$ is bijective.*

As a corollary of Theorem 1.3, we have the following.

Corollary 1.4. *Let $\eta = (E, p, Y)$ be a definable G-vector bundle over a definable G-set Y and $f, h : X \to Y$ definable G-maps between definable G-sets. If f and h are G-homotopic, then $f^*(\eta)$ and $h^*(\eta)$ are definably G-isomorphic.*

Let $1 \le r \le \omega$. A *definable C^rG-manifold* is a pair (X, θ) consisting of a definable C^r-manifold X and a group action $\theta : G \times X \to X$ which is a definable C^r-map. We simply write X for (X, θ). A definable C^rG-manifold is *affine* if it is definably C^rG-diffeomorphic to a G-invariant definable C^r-submanifold of some representation of G.

Two definable C^rG-maps $f, h : X \to Y$ between definable C^rG-manifolds are *definably C^rG-homotopic* if there exists a definable C^rG-map $H : X \times [0,1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = h(x)$ for all $x \in X$, where G acts on $[0, 1]$ trivially.

The following result is a definable C^rG-version of Theorem 1.1.

Theorem 1.5. *Suppose that $\eta = (E, p, Y, F, K)$ is a definable C^rG-fiber bundle over a definable C^rG-manifold Y and $1 \le r < \infty$. Let f, h be definable C^rG-maps from a compact affine definable C^rG-manifold X to Y. If f and h are definably C^rG-homotopic and F is affine, then $f^*(\eta)$ and $h^*(\eta)$ are definably C^rG-isomorphic.*

Corollary 1.6. *Let $f, h : X \to Y$ be definable C^rG-maps between definable C^rG-manifolds and $1 \le r < \infty$. If X is compact and affine, η is a definable C^rG-vector bundle over Y and f is definably C^rG-homotopic to h, then $f^*(\eta)$ and $h^*(\eta)$ are definably C^rG-isomorphic.*

Let $1 \le r \le \omega$. A definable C^rG-vector bundle $\eta = (E, p, X)$ over an affine definable C^rG-manifold X is called *strongly definable* if then there exist a representation Ω of G and a definable C^rG-map $f : X \to G(\Omega, k)$ such that η is definably C^rG-isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η.

Theorem 1.7. *Let η be a definable C^rG-vector bundle over an affine definable C^rG-manifold X. If X is compact and $1 \le r < \infty$, then η is strongly definable. Moreover if $r = \infty$ or ω, then η is strongly definable if and only if the total space of η is affine.*

This paper is organized as follows. In section 2, we give a definition of definable G fiber bundles and prove Theorem 1.1. We prove Theorem 1.2, 1.3 and Corollary 1.4 in section 3 and Theorem 1.5 and 1.7 in section 4.

2. Definable G-fiber bundles

A group homomorphism between definable groups is a *definable group homomorphism* if it is a definable map. An *n-dimensional representation* of a definable group G means $\mathbb{R}^n$ with the linear action induced by a definable group homomorphism from G to $O_n(\mathbb{R})$. A subgroup of a definable group G is a *definable subgroup* of G if it is a definable subset of G. A definable map (respectively A definable homeomorphism) between definable G-sets is a *definable G-map* (respectively a *definable G-homeomorphism*) if it is a G-map.

Let G be a definable group. A *definable set with a definable G-action* is a pair (X, θ) consisting of a definable set X and a group action $\theta : G \times X \to X$ such that θ is a definable map. We simply write X instead of (X, θ). This action is not necessarily linear (orthogonal). *Definable G-maps* and *definable G-homeomorphisms* between definable sets with definable G-actions are defined similarly.

A *definable space* is an object obtained by pasting finitely many definable sets together along open definable subsets, and definable maps between definable spaces are defined similarly (see Chapter 10 [6]). Definable spaces are generalizations of semialgebraic spaces in the sense of [4].

Definition 2.1. Let G be a definable group.

(1) A *definable G-space* is a pair (X, θ) consisting of a definable space X and a group action $\theta : G \times X \to X$ which is definable. For simplicity of notation, we write X for (X, θ).

(2) Let X and Y be definable G-spaces. A definable map $f : X \to Y$ is called a *definable G-map* if it is a G-map. We say that X and Y are *definably G-homeomorphic* if there exist definable G-maps $h : X \to Y$ and $k : Y \to X$ such that $h \circ k = id$ and $k \circ h = id$.

Note that clearly an implication "a definable G-set" $\Rightarrow$ "a definable set with a definable G-action" $\Rightarrow$ "a definable G-space" holds.

Definition 2.2. (1) A topological fiber bundle $\eta = (E, p, X, F, K)$ is called a *definable fiber bundle* over X with fiber F and structure group K if the following two conditions are satisfied:

(a) The total space E is a definable space, the base space X is a definable set, the structure group K is a definable group, the fiber F is a definable set with an effective definable K action, and the projection $p : E \to X$ is a definable map.

(b) There exists a finite family of local trivializations $\{U_i, \phi_i : p^{-1}(U_i) \to U_i \times F\}_i$ of η such that each U_i is a definable open subset of X, $\{U_i\}_i$ is a finite open covering of X. For any $x \in U_i$, let $\phi_{i,x} : p^{-1}(x) \to F, \phi_{i,x}(z) = \pi_i \circ \phi_i(z)$, where

π_i stands for the projection $U_i \times F \to F$. For any i and j with $U_i \cap U_j \neq \emptyset$, the transition function $\theta_{ij} := \phi_{j,x} \circ \phi_{i,x}^{-1} : U_i \cap U_j \to K$ is a definable map. We call these trivializations *definable*.

Definable fiber bundles with compatible definable local trivializations are identified.

(2) Let $\eta = (E, p, X, F, K)$ and $\zeta = (E', p', X', F, K)$ be definable fiber bundles whose definable local trivializations are $\{U_i, \phi_i\}_i$ and $\{V_j, \psi_j\}_j$, respectively. A definable map $\overline{f} : E \to E'$ is said to be a *definable morphism* if the following two conditions are satisfied:

(a) The map $\overline{f}$ covers a definable map, namely there exists a definable map $f : X \to X'$ such that $f \circ p = p' \circ \overline{f}$.

(b) For any i, j such that $U_i \cap f^{-1}(V_j) \neq \emptyset$ and for any $x \in U_i \cap f^{-1}(V_j)$, the map $f_{ij}(x) := \psi_{j,f(x)} \circ \overline{f} \circ \phi_{i,x}^{-1} : F \to F$ lies in K, and $f_{ij} : U_i \cap f^{-1}(V_j) \to K$ is a definable map.

We say that a bijective definable morphism $\overline{f} : E \to E'$ is a *definable equivalence* if it covers a definable homeomorphism $f : X \to X'$ and $(\overline{f})^{-1} : E' \to E$ is a definable morphism covering $f^{-1} : X' \to X$. A definable equivalence $\overline{f} : E \to E'$ is called a *definable isomorphism* if $X = X'$ and $f = id_X$.

(3) A continuous section $s : X \to E$ of a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *definable section* if for any i, the map $\phi_i \circ s|U_i : U_i \to U_i \times F$ is a definable map.

(4) We say that a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *principal definable fiber bundle* if $F = K$ and the K-action on F is defined by the multiplication of K. We write (E, p, X, K) for (E, p, X, F, K).

Definition 2.3. Let G be a definable group.

(1) A definable fiber bundle (E, p, X, F, K) (respectively A principal definable fiber bundle (E, p, X, K)) is called a *definable G-fiber bundle* (respectively a *principal definable G-fiber bundle*) if the total space E is a definable G-space such that G acts on E through definable equivalences, the base space X is a definable set with a definable G-action and the projection p is a definable G-map.

(2) A definable morphism (respectively A definable equivalence, A definable isomorphism) between definable G-fiber bundles is a *definable G-morphism* (respectively a *definable G-equivalence*, a *definable G-isomorphism*) if it is a G-map.

(3) A *definable G-section* of a definable G-fiber bundle means a definable section which is a G-map.

Let $f : X \to Y$ be a definable map between definable sets. We say that f is *proper* if for any compact subset C of Y, $f^{-1}(C)$ is compact.

Let E be an equivalence relation on a definable set X. We call E *proper* if E is a definable subset of $X \times X$ and the projection $E \to X$ defined by $(x, y) \mapsto x$ is proper.

Theorem 2.4 (Definable quotients (e.g. 10.2.15 [6])). *Let E be a proper equivalence relation on a definable set X. Then X/E exists a proper quotient, namely X/E is a definable subset of some $\mathbb{R}^n$ and the projection $X \to X/E$ is a surjective proper definable map.*

In the remainder of this section, G and K denote compact definable groups. The following is a corollary of Theorem 2.4.

Corollary 2.5 (e.g. 10.2.18 [6]). *Let X be a definable set with a definable G-action. Then X/G is a definable subset of some $\mathbb{R}^n$ and the orbit map $p : X \to X/G$ is a surjective proper definable map.*

By similar proofs of 2.10 [14] and 2.11 [14], the standard construction of the associated principal bundle from a fiber bundle and by Theorem 2.4, we have the following.

Proposition 2.6. (1) *Let (E, p, X, K) be a principal definable G-fiber bundle and F a definable set with an effective definable K-action. Then $(E \times_K F, p', X, F, K)$ is a definable G-fiber bundle, where $p' : E \times_K F \to X$ denotes the projection defined by $p'([z, k]) = p(z)$.*

(2) *The associated principal G-fiber bundle of a definable G-fiber bundle is definable.*

(3) *Two definable G-fiber bundles having the same base space, fiber and structure group are definably G-isomorphic if and only if their associated principal definable G-fiber bundles are definably G-isomorphic.*

Let X be a definable set with a definable G-action and $x \in X$. A G_x-invariant definable subset S of X is a *definable slice* at x in X if GS is a G-invariant definable open neighborhood of the orbit $G(x)$ of x in X, $G \times_{G_x} S$ is a definable set with the standard definable G-action $G \times (G \times_{G_x} S) \to G \times_{G_x} S, (g, [g', s]) \mapsto [gg', s]$, and the map $G \times_{G_x} S \to GS \subset X$ defined by $[g, s] \mapsto gs$ is a definable G-homeomorphism.

Theorem 2.7 (Definable slices). *Let X be a definable G-set and $x \in X$. Then there exists a definable slice S at x in X.*

Let Y be a G-invariant definable subset of a definable G-set X. A *definable G-retraction from X to Y* means a definable G-map $R : X \to Y$ with $R|Y = id_Y$.

For the proof of Theorem 2.7, we recall the following result.

Theorem 2.8 (3.4 [11]). *Let Y be a G-invariant definable closed subset of a definable G-set X. Then there exist a G-invariant definable open neighborhood U of Y in X and a definable G-retraction from U to Y.*

Proof of Theorem 2.7. Since $G(x)$ is a G-invariant definable closed subset of X and by Theorem 2.8, we have a G-invariant definable open neighborhood U of $G(x)$ in X and a definable G-retraction q from U to $G(x)$. Let $S := q^{-1}(x)$. Then S is a definable G_x-set and $U = GS$. By II.4.2 [2], the map $f : G \times_{G_x} S \to GS$ $(\subset X)$ defined by $f([g, s]) = gs$ is a G-homeomorphism. On the other hand, the map $k : G \times S \to GS$ defined by $k(g, s) = gs$ and the projection $\pi : G \times S \to G \times_{G_x} S$ are definable maps. Since the graph of f is the image of that of k by $\pi \times id_{GS}$, f is a definable G-homeomorphism. □

Definition 2.9. A definable G-fiber bundle $\eta = (E, p, X, F, K)$ satisfies the *definable Bierstone condition* if for any $x \in X$, there exist a G_x-invariant definable open neighborhood U_x of x in X and a definable group homomorphism $\rho_x : G_x \to K$ such that $\eta|U_x$ is definably G_x-isomorphic to $U_x \times F$ with the definable G_x-action defined by $G_x \times (U_x \times F) \to U_x \times F, (h, u, y) \mapsto (hu, \rho_x(h)y)$.

Note that a definable G-fiber bundle over a definable G-set satisfies the definable Bierstone condition if and only if the associated principal definable G-fiber bundle satisfies it.

Using Theorem 2.7, similar proofs of 1.4 [15] and 1.5 [15] prove the following proposition.

Proposition 2.10. *Every definable G-fiber bundle over a definable G-set satisfies the definable Bierstone condition.*

A finite definable open covering $\{U_i\}_i$ of a definable G-set is called a *finite definable open G-covering* if each U_i is G-invariant. A finite definable G-open covering is *numerable* if there exists a definable partition of unity $\{\lambda_i\}_i$ subordinate to $\{U_i\}_i$ such that each λ_i is G-invariant.

The following proposition shows existence of (non-equivariant) definable partition of unity.

Proposition 2.11 (e.g. 6.3.7 [6]). *Let X be a definable set in $\mathbb{R}^n$ and $\{U_i\}_{i=1}^n$ a finite definable open covering of X. Then there exists a definable partition of unity subordinate to $\{U_i\}_{i=1}^n$, namely there exist definable functions $\lambda_1, \dots, \lambda_n : X \to \mathbb{R}$ such that $0 \le \lambda_i \le 1$, supp $\lambda_i \subset U_i$ and $\sum_{i=1}^n \lambda_i = 1$.*

The following is an equivariant version of Proposition 2.11.

Proposition 2.12 (Equivariant definable partition of unity). *Every finite definable open G-covering of a definable G-set X is numerable.*

Proof. Let $\{U_i\}_{i=1}^n$ be a finite definable open G-covering of a definable G-set X. By Corollary 2.5, the orbit map $p : X \to X/G$ is a surjective proper definable map. Since $p : X \to X/G$ is open, $\{p(U_i)\}_{i=1}^n$ is a finite definable open covering of X/G. By Proposition 2.11, one can find a definable partition of unity $\{\overline{\lambda}_i\}_{i=1}^n$ subordinate to $\{p(U_i)\}_{i=1}^n$. Hence $\lambda_1 := \overline{\lambda}_1 \circ p, \dots, \lambda_n := \overline{\lambda}_n \circ p$ are G-invariant and subordinate to $\{U_i\}_{i=1}^n$. □

Note that in Proposition 2.11 and 2.12, we can replace $\sum_{i=1}^n \lambda_i = 1$ by $\max_{1 \le i \le n} \lambda_i = 1$.

Theorem 1.1 follows from Theorem 2.13 below.

Theorem 2.13. *If X is a compact definable G-set, then every definable G-fiber bundle $\eta = (E, p, X \times [0,1], F, K)$ is definably G-isomorphic to $(p^{-1}(X \times \{0\}) \times [0,1], p', X \times [0,1], F, K)$, where G acts on $[0,1]$ trivially, $X \times \{0\}$ is identified with X and $p' = p|p^{-1}(X \times \{0\}) \times id_{[0,1]}$.*

To prove Theorem 2.13, we need the following three results.

Lemma 2.14. *Let A be a definable G-set, $X_1 = A \times [a,b], X_2 = A \times [b,c]$, and $\eta = (E, p, X, F, K)$ a definable G-fiber bundle over $X = X_1 \cup X_2$, where G acts trivially on $[a,b]$ and $[b,c]$. If $\eta|X_1$ and $\eta|X_2$ are definably G-isomorphic to $X_1 \times F$ and $X_2 \times F$, respectively, then so is η, where the action on F is induced by a definable group homomorphism from G to K.*

Proof. Let $u_i : X_i \times F \to p^{-1}(X_i)$, $(i = 1,2)$, be definable G-isomorphisms and $w_i := u_i|(X_1 \cap X_2) \times F$, $(i = 1,2)$. Then $h := w_2^{-1} \circ w_1 : (X_1 \cap X_2) \times F \to (X_1 \cap X_2) \times F$

is a definable G-isomorphism. Hence there exists a definable map $l : X_1 \cap X_2 \to K$ such that $h(x,y) = (x, l(x)y)$, where $(x,y) \in (X_1 \cap X_2) \times F$. Let $i_A : A \to K, i_A(a) = l(a,b)$. Then we can extend h to a definable G-isomorphism

$$\tilde{h} : X_2 \times F \to X_2 \times F, \tilde{h}(x_1, x_2, y) = (x_1, x_2, i_A(x_1)y).$$

Since two definable G-isomorphisms $u_1 : X_1 \times F \to p^{-1}(X_1)$ and $u_2 \circ \tilde{h} : X_2 \times F \to p^{-1}(X_2)$ coincide on $(X_1 \cap X_2) \times F$ and $X_1 \times F$ and $X_2 \times F$ are closed in $(X_1 \cup X_2) \times F = X \times F$, the gluing map provides the required definable G-isomorphism. □

Let H be a definable subgroup of G, $\rho : H \to K$ a definable group homomorphism between definable groups, and F a definable set with an effective definable K-action. For any definable H-set S, we define a definable G-fiber bundle $\epsilon^\rho(S)$ by $(G \times_H (S \times F), p, G \times_H S, F, K)$, where $p : G \times_H (S \times F) \to G \times_H S, p([g,(s,y)]) = [g,s]$ and H acts on F via ρ.

Lemma 2.15. *Let X be a compact definable G-set and $\eta = (E, p, X \times [0,1], F, K)$ a definable G-fiber bundle over $X \times [0,1]$. Then there exist finitely many points $x_1, \dots x_n$ with definable slices $S_{x_1}, \dots, S_{x_n}$ and definable group homomorphisms $\{\rho_i : G_{x_i} \to K\}_{i=1}^n$ such that $\{GS_{x_i}\}_{i=1}^n$ is a finite definable open G-covering of X and each $\eta|(GS_{x_i} \times [0,1])$ is definably G-equivalent to $\epsilon^{\rho_i}(S_{x_i}) \times [0,1]$.*

Proof. By Proposition 2.10, for any $(x,t) \in X \times [0,1]$, there exist a G_x-invariant definable open neighborhood U_x of x in X and $\delta > 0$ such that $\eta|(U_x \times [t-\delta, t+\delta])$ is definably G_x-isomorphic to $(U_x \times [t-\delta, t+\delta]) \times F$, where the action on F is induced by a definable group homomorphism $\rho_x : G_x \to K$. Since $[0,1]$ is compact and by Lemma 2.14, we have a G_x-invariant definable open neighborhood V_x of x in X such that $\eta|V_x \times [0,1]$ is definably G_x-isomorphic to $(V_x \times [0,1]) \times F$. By Theorem 2.7, we have a definable slice S_x at x with $S_x \subset V_x$. Hence there exists a definable G_x-isomorphism $l_x : S_x \times [0,1] \times F \to \eta|S_x \times [0,1]$. Thus $h_x : G \times_{G_x} (S_x \times [0,1] \times F) = \epsilon^{\rho_x}(S_x) \times [0,1] \to \eta|GS_x \times [0,1]$ defined by $h_x([g,(s,t,f)]) = gl_x(s,t,f)$ is a definable G-equivalence. Since X is compact, there exist finitely many points $x_1, \dots, x_n$ of X such that $\{GS_{x_i}\}_{i=1}^n$ is a finite definable open G-covering of X. □

Theorem 2.16. *Let X be a compact definable G-set, $r : X \times [0,1] \to X \times [0,1], r(x,t) = (x,1)$ and $\eta = (E, p, X \times [0,1], F, K)$ a definable G-fiber bundle over $X \times [0,1]$. Then there exists a definable G-morphism $\phi : E \to E$ covering r.*

Proof. By Lemma 2.15, we can find finitely many points $x_1, \dots, x_n$ with definable slices $S_{x_1}, \dots, S_{x_n}$ and definable group homomorphisms $\{\rho_i : G_{x_i} \to K\}_{i=1}^n$ such that $\{GS_{x_i}\}_{i=1}^n$ is a finite definable open G-covering of X and each $\eta|(GS_{x_i} \times [0,1])$ is definably G-equivalent to $\epsilon^{\rho_{x_i}}(S_{x_i}) \times [0,1]$. By Proposition 2.12, there exist G-invariant definable functions $l_1, \dots, l_n : X \to [0,1]$ such that:

(a) The support of each l_i is contained in GS_{x_i}.
(b) $\max_{1 \le i \le n} l_i(x) = 1$ for all $x \in X$.

Let $h_{x_i} : (G \times_{G_{x_i}} (S_{x_i} \times F)) \times [0,1] \to p^{-1}(GS_{x_i} \times [0,1])$ be a definable G-equivalence covering a definable G-homeomorphism $f_{x_i} \times id_{[0,1]} : (G \times_{G_{x_i}} S_{x_i}) \times [0,1] \to GS_{x_i} \times [0,1]$.

Define

$$(u_i, r_i) : (E, X \times [0,1]) \to (E, X \times [0,1]), 1 \le i \le n,$$

$$r_i(x,t) = \begin{cases} (x, \max(l_i(f_{x_i}([g,s])), t)), & ([g,s],t) \in (G \times_{G_{x_i}} S_{x_i}) \times [0,1] \\ (x,t), & \text{otherwise} \end{cases},$$

$$u_i(h_{x_i}([g,(s,f)],t) = h_{x_i}([g.(s,f)], \max(l_i(f_{x_i}([g,s])),t)),$$
$$\text{for any } ([g,(s.f)],t) \in (G \times_{G_{x_i}} (S_{x_i} \times F)) \times [0,1],$$

u_i is the identity outside $p^{-1}(GS_{x_i} \times [0,1])$.

Then $r = r_n \circ \cdots \circ r_1$. Therefore $\phi = u_n \circ \cdots \circ u_1 : E \to E$ is the required definable G-morphism. □

Theorem 2.13 follows from Theorem 2.16.

3. Definable G-vector bundles and proof of Theorem 1.2, 1.3 and Corollary 1.4

We recall that G and K denote compact subgroups of $GL_n(\mathbb{R})$ except section 2. Then remember that G is a compact algebraic subgroup of $GL_n(\mathbb{R})$ and any closed subgroup of G is a compact algebraic subgroup of G.

Note that a definable group homomorphism from G to $O_n(\mathbb{R})$ is a definable C^∞-map because it is a continuous group homomorphism between Lie groups.

Recall universal G-vector bundles (e.g. [12]).

Definition 3.1. Let Ω be an n-dimensional representation of G induced by a definable group homomorphism $B : G \to O_n(\mathbb{R})$ of Ω. Suppose that $M(\Omega)$ denotes the vector space of $n \times n$-matrices with the action $(g, A) \in G \times M(\Omega) \to B(g)AB(g)^{-1} \in M(\Omega)$. For any positive integer k, we define the vector bundle $\gamma(\Omega, k) = (E(\Omega,k), u, G(\Omega,k))$ as follows:

$$G(\Omega,k) = \{A \in M(\Omega) | A^2 = A, A = A', Tr A = k\},$$
$$E(\Omega,k) = \{(A,v) \in G(\Omega,k) \times \Omega | Av = v\},$$
$$u : E(\Omega,k) \to G(\Omega,k), u((A,v)) = A,$$

where A' denotes the transposed matrix of A and $Tr\ A$ stands for the trace of A. Then $\gamma(\Omega,k)$ is an algebraic vector bundle. Since the action on $\gamma(\Omega,k)$ is algebraic, it is an algebraic G-vector bundle. We call it *the universal G-vector bundle associated with Ω and k*. Remark that $G(\Omega,k) \subset M(\Omega)$ and $E(\Omega,k) \subset M(\Omega) \times \Omega$ are nonsingular algebraic G-sets.

Definition 3.2. (1) A *definable G-vector bundle of rank k* is a definable G-fiber bundle with fiber $\mathbb{R}^k$ and structure group $GL_k(\mathbb{R})$. We usually write (E,p,X) instead of $(E,p,X,\mathbb{R}^k, GL_k(\mathbb{R}))$.

(2) Let $\eta = (E,p,X)$ and $\eta' = (E',p',X)$ be definable G-vector bundles. A definable G-map $f : E \to E'$ is called a *definable G-morphism* if $p = p' \circ f$ and f is linear on each fiber. A definable G-morphism $h : E \to E'$ is said to be a *definable G-isomorphism* if there exists a definable G-morphism $h' : E' \to E$ such that $h \circ h' = id$ and $h' \circ h = id$.

(3) A *definable G-section* of a definable G-vector bundle means a definable G-section as a definable G-fiber bundle.

By a way similar to 3.1 [10], we have the following proposition.

Proposition 3.3. *If η and η' are two definable G-vector bundles over a definable G-set X, then $\eta \oplus \eta'$, $\eta \otimes \eta'$, $Hom(\eta, \eta')$ and the dual bundle $\eta^\vee$ of η are definable G-vector bundles over X.*

The next result states equivalent properties of strong definablity of definable G vector bundles, which is obtained in a way similar to the proof of 3.6 [3].

Theorem 3.4. *Let $\eta = (E, p, X)$ be a definable G-vector bundle of rank k over a definable G-set X. Then the following five properties are equivalent.*

(1) *The bundle η is strongly definable.*
(2) *There exists a surjective definable G-morphism from a trivial G-vector bundle $X \times \Omega$ onto η for some representation Ω of G.*
(3) *There exists an injective definable G-morphism from η to a trivial G-vector bundle $X \times \Omega$ for some representation Ω of G.*
(4) *There exists a definable G-vector bundle η' over X such that $\eta \oplus \eta'$ is definably G-isomorphic to a trivial G-vector bundle.*
(5) *There exist non-equivariant definable sections $s_1, \dots, s_n : X \to E$ of η such that:*
   (a) *For any $x \in X$, the vectors $s_1(x), \dots, s_n(x)$ generate the fiber $p^{-1}(x)$ over x.*
   (b) *The sections $s_1, \dots, s_n$ generate a finite dimensional G-invariant vector subspace of $\Gamma(\eta)$, where $\Gamma(\eta)$ denotes the set of all continuous sections of η with the natural G-action, namely $(g \cdot s)(x) = g(s(g^{-1}x))$ for all $g \in G$ and $x \in X$.*

Theorem 1.2 follows from Theorem 3.4 and Theorem 3.5 below.

Theorem 3.5. *Every definable G vector bundle over a definable G set satisfies Condition (5) in Theorem 3.4.*

By a way similar to the proof of 3.9 [3], we have the following proposition.

Proposition 3.6. *Let $\eta = (E, p, X)$ be a definable G-vector bundle over a definable set X with the trivial G-action and A a closed definable subset of X such that $\eta|A$ is strongly definable. If A admits a definable retraction from X to A, then there exists some open definable neighborhood V of A in X such that $\eta|V$ is strongly definable.*

The following is the equivariant definable version of Urysohn's lemma, and its semialgebraic version is proved in 1.6 [5]. We use only a non-equivariant version of it to prove Theorem 3.5.

Lemma 3.7. *Let X be a definable set with a definable G-action and A and B disjoint closed definable G-subsets of X. Then there exists a G-invariant definable function $f : X \to [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.*

Proof. By Corollary 2.5, X/G is a definable subset of some $\mathbb{R}^n$ and the orbit map $p : X \to X/G$ is a surjective proper definable map. Hence $\pi(A)$ and $\pi(B)$ are closed definable subsets of X/G. Then the function $h : X/G \to [0,1]$ defined by $h(x) = \frac{d(x,\pi(A))}{d(x,\pi(A))+d(x,\pi(B))}$ is a definable function such that $h^{-1}(0) = \pi(A)$ and $h^{-1}(1) = \pi(B)$, where $d(x,\pi(A))$ (respectively $d(x,\pi(B))$) denotes the distance between x and $\pi(A)$ (respectively x and $\pi(B)$)). Therefore $f := h \circ \pi : X \to [0,1]$ is the required G-invariant definable function. □

Proposition 3.8. *Let H be a closed subgroup of G, D the closed unit ball of a representation Ω of H. Then $G \times_H D$ is a compact affine definable $C^\infty G$ manifold with boundary. In particular, $G \times_H D$ is definably G-imbeddable into some representation of G.*

Proof. Note that G and Ω are affine definable $C^\infty H$-manifolds. Thus by 4.4 [13] and 4.5 [13], $G \times_H \Omega$ is a definable $C^\infty G$-manifold whose underlying manifold is a definable C^∞-submanifold of some $\mathbb{R}^k$. Since $G \times_H D$ is compact, there exists a $C^\infty G$-imbedding i from $G \times_H D$ to some representation Ξ of G. Applying the polynomial approximation theorem to i and averaging it, we have a definable $C^\infty G$-imbedding from $G \times_H D$ to Ξ. □

A *definable G-CW-complex* is a finite G-CW-complex such that the characteristic map of each G-cell is a definable G-map (see [11]).

Theorem 3.9 (1.1 [11]). *Let X be a definable G-set and Y a closed definable G-subset of X. Then there exist a definable G-CW-complex Z in a representation Ω of G, a G-CW-subcomplex W of Z, and a definable G-map $f : X \to Z$ such that:*

1. *The map f takes X and Y definably G-homeomorphically onto G-invariant definable subsets Z_1 and W_1 of Z and W obtained by removing some open G-cells from Z and W, respectively.*
2. *The orbit map $\pi : Z \to Z/G$ is a definable cellular map.*
3. *The orbit space Z/G is a finite simplicial complex compatible with $\pi(Z_1)$ and $\pi(W_1)$.*
4. *For each open G-cell c of Z, $\pi|\bar{c} : \bar{c} \to \pi(\bar{c})$ has a definable section $s : \pi(\bar{c}) \to \bar{c}$, where $\bar{c}$ denotes the closure of c in Z.*

Furthermore, if X is compact, then $Z = f(X)$ and $W = f(Y)$.

Using Proposition 3.6, Lemma 3.7, Proposition 3.8, Theorem 3.9, a similar proof of 3.5 [3] proves Theorem 3.5.

By Theorem 1.2 and by the proof of 4.7 [11], we have the following.

Proposition 3.10. *Let η a definable G-vector bundle over a compact definable G-set X. Then every continuous G-section of η can be approximated by definable G-sections.*

We obtain the following theorem using Proposition 3.3 and Proposition 3.10.

Theorem 3.11. *Let η and ζ be definable G-vector bundles over a compact definable G-set. If η is G-isomorphic to ζ, then they are definably G-isomorphic.*

Proposition 3.12 (2.11 [15]). *Let X, Y be definable G-sets. If η is G-vector bundle over Y and $f, h : X \to Y$ are G-homotopic continuous G-maps, then $f^*(\eta)$ is G-isomorphic to*

Proposition 3.13 ([1], [20]). *Let X be a compact G-set. If η is a G-vector bundle over X, then there exist a representation Ω of G and a continuous G-map $f : X \to G(\Omega, k)$ such that η is G-isomorphic to $f^*(\gamma(\Omega, k))$. where k denotes the rank of η.*

Theorem 3.14. *If X is a compact definable G-set, $\kappa : Vect_G^{def}(X) \to Vect_G(X)$ is bijective.*

Proof. Injectivity follows from Theorem 3.11.

Let η be a G-vector bundle over X. Then by Proposition 3.13, there exist a representation Ω of G and a continuous G-map $f : X \to G(\Omega, k)$ such that η is G-isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η. By 3.5 [11], f is G-homotopic to a definable G-map $h : X \to G(\Omega, k)$. Hence by Proposition 3.12, $f^*(\gamma(\Omega, k))$ is G-isomorphic to $h^*(\gamma(\Omega, k))$. Therefore η is G-isomorphic to a definable G-vector bundle $h^*(\gamma(\Omega, k))$. □

A G-set X is *G-contractible* if there exist a fixed point $x_0 \in X$ and a continuous G-map $F : X \times [0, 1] \to X$ such that $F(x, 0) = x$ and $F(x, 1) = x_0$ for all $x \in X$, where G acts on $[0, 1]$ trivially. We have the following as a corollary of Theorem 1.1.

Corollary 3.15. *Let X be a compact G-contractible definable G-set. Then every definable G-vector bundle over X is definably G-isomorphic to a trivial G-bundle.*

Theorem 3.16 (3.3 [11]). *Let X be a definable G-set. Then there exists a definable G-deformation retraction R from X to a compact definable G-subset Y of X.*

By a way similar to the proof of 4.10 [11], we have the following proposition.

Proposition 3.17. *The map $R^* : Vect_G^{def}(Y) \to Vect_G^{def}(X)$ defined by $\eta \mapsto R^*(\eta)$ is bijective.*

Theorem 1.3 follows from Theorem 3.14 and Proposition 3.17. Corollary 1.4 follows from Theorem 1.3 and Proposition 3.12.

4. DEFINABLE C^rG-FIBER BUNDLES AND DEFINABLE C^rG-VECTOR BUNDLES

Definition 4.1 ([12]). Let $1 \leq r \leq \omega$.

(1) A definable fiber bundle $\eta = (E, p, X, F, K)$ is a *definable C^r-fiber bundle* if the total space E and the base space X are definable C^r-manifolds, the structure group K is a definable C^r-group, the fiber F is a definable C^rK-manifold with an effective action, the projection p is a definable C^r-map and all transition functions of η are definable C^r-maps. A *principal definable C^r-fiber bundle* is defined similarly.

(2) *Definable C^r-morphisms*, *definable C^r-equivalences*, *definable C^r-isomorphisms* between definable C^r-fiber bundles and *definable C^r-sections* of a definable C^r fiber bundle are defined similarly.

(3) A definable C^r-fiber bundle $\eta = (E, p, X, F, K)$ is a *definable C^rG-fiber bundle* if the total space E and the base space X are definable C^rG-manifolds, the projection p is a definable C^rG-map and G acts on E through definable C^r-equivalences. A *principal definable C^rG-fiber bundle* is defined similarly.

(4) A definable C^r-morphism (resp. a definable C^r-equivalence, a definable C^r-isomorphism, a definable C^r-section) is a *definable C^rG-morphism* (resp. a *definable C^rG-equivalence*, a *definable C^rG-isomorphism*, a *definable C^rG-section*) if it is a G-map.

The following is a definable C^rG-version of Proposition 2.6, which is obtained similarly.

Proposition 4.2. *Suppose that $1 \leq r \leq \omega$.*

(1) *Let (E, p, X, K) be a principal definable C^rG-fiber bundle and F an affine definable C^rK-manifolds with an effective action. Then $(E \times_K F, p', X, F, K)$ is a definable C^rG-fiber bundle, where $p' : E \times_K F \to X$ denotes the projection defined by $p'([z, k]) = p(z)$.*
(2) *The associated principal G-fiber bundle of a definable C^rG-fiber bundle is a principal definable C^rG-fiber bundle.*
(3) *Two definable C^rG-fiber bundles having the same base space, fiber and structure group are definably C^rG-isomorphic if and only if their associated principal definable C^rG-fiber bundles are definably C^rG-isomorphic.*

Proposition 4.3. *Let X be a definable C^rG-submanifold of a representation Ω of G and $1 \leq r < \infty$. Then for any $x \in X$, there exists a linear definable C^r-slice at x in X, namely there exists a definable C^rG_x-imbedding i from a representation Ξ of G_x into X such that $i(0) = x$, $G \times_{G_x} \Xi$ is a definable C^rG-manifold with the standard action $(g, [g', x]) \mapsto [gg', x]$ and the map $\mu : G \times_{G_x} \Xi \to X$ defined by $[g, x] \mapsto gi(x)$ is a definable C^rG-diffeomorphism onto some G-invariant definable open neighborhood of $G(x)$ in X.*

Proof. Since G is a compact algebraic subgroup of $GL_n(\mathbb{R})$ and by 4.1 [13], for any $x \in X$, there exists a linear definable C^∞ slice at x in Ω, namely we have a representation Ξ' of G_x and a definable $C^\infty G_x$ imbedding $j : \Xi' \to \Omega$ such that $j(0) = x$, $G \times_{G_x} \Xi'$ is a definable $C^\infty G$ manifold and the map $\mu' : G \times_{G_x} \Xi' \to \Omega$ defined by $\mu'([g, x]) = gj(x)$ is a definable $C^\infty G$ diffeomorphism onto a G invariant definable open neighborhood $Gj(\Xi')$ of $G(x)$ in Ω. Then $j^{-1}(X)$ is a definable C^rG_x submanifold of Ξ' and $j|j^{-1}(X) : j^{-1}(X) \to X$ is a definable C^rG_x imbedding. Hence there exists a sufficiently small G_x invariant definable open neighborhood U of 0 in $j^{-1}(X)$ such that U is definably C^rG_x diffeomorphic to a representation Ξ of G_x. Take a definable C^rG_x diffeomorphism $l : \Xi \to U$ with $l(0) = 0$ and let $i = j \circ l$. Then i is a definable C^rG_x imbedding from Ξ to X and the map $\mu : G \times_{G_x} \Xi \to X$ defined by $\mu([g, x]) = gi(x)$ is a definable C^rG diffeomorphism onto a G invariant definable open neighborhood $Gi(\Xi) = Gj(U)$ of $G(x)$ in X. □

Note that if $r = \infty$ or ω, then Proposition 4.3 is proved in 4.1 [13].

We can consider the *definably C^r-Bierstone condition* as a definable C^rG-version of Definition 2.9. Using Proposition 4.2 and 4.3, we have the following definable C^r-version of Proposition 2.10.

Proposition 4.4. *Let $1 \leq r \leq \omega$. Then every definable C^rG-fiber bundle over an affine definable C^rG-manifold satisfies the definable C^r-Bierstone condition.*

The proof of 4.8 [12] proves the following.

Proposition 4.5 (4.8 [12]). *(Definable C^r partition of unity). Let X be a definable closed subset of $\mathbb{R}^n$, $\{U_i\}_{i=1}^l$ a finite definable open covering of X and $0 \leq r < \infty$. Then there exist definable C^r functions $\lambda_1, \dots, \lambda_l : \mathbb{R}^n \to \mathbb{R}$ such that $0 \leq \lambda_i \leq 1$, supp $\lambda_i \subset U_i$ and $\sum_{i=1}^l \lambda_i(x) = 1$ for any $x \in X$.*

The following is a definable C^r-version of Proposition 2.12.

Proposition 4.6 (Equivariant definable C^r-partition of unity). *Let X be a definable C^rG-submanifold closed in a representation Ω of G and $\{U_i\}_{i=1}^n$ a finite definable open G-covering of X and $0 \leq r < \infty$. Then $\{U_i\}_{i=1}^n$ is numerable, namely there exist G-invariant definable C^r-functions $\lambda_1, \dots, \lambda_n : X \to \mathbb{R}$ such that $0 \leq \lambda_i \leq 1$, supp $\lambda_i \subset U_i$ and $\sum_{i=1}^n \lambda_i(x) = 1$ for any $x \in X$.*

Proof. First of all, we recall the structure of the orbit space Ω/G. The algebra $\mathbb{R}[\Omega]^G$ of G invariant polynomials on Ω is finitely generated [21]. Let $p_1, \dots, p_n : \Omega \to \mathbb{R}$ be G invariant polynomials generating $\mathbb{R}[\Omega]^G$, and put $p : \Omega \to \mathbb{R}^n, p = (p_1, \dots, p_n)$. Then p is a proper polynomial map, and it induces a closed imbedding $j : \Omega/G \to \mathbb{R}^n$ such that $p = j \circ \pi$, where $\pi : \Omega \to \Omega/G$ denotes the orbit map. Hence we can identify Ω/G (resp. X/G, π) with $j(\Omega/G)$ (resp. $j(X/G)$, p). Thus $\{p(U_i)\}_{i=1}^l$ is a finite definable open covering of X/G because $p|X : X \to X/G$ is open. Note that $p(X)$ is closed in $\mathbb{R}^n$ because X is closed in Ω. By Proposition 4.5, one can find a definable partition of unity $\{\overline{\lambda}_i\}_{i=1}^l$ subordinate to $\{p(U_i)\}_{i=1}^l$. Hence $\lambda_1 := \overline{\lambda}_1 \circ p, \dots, \lambda_l := \overline{\lambda}_l \circ p$ are the required G invariant definable C^r functions. $\square$

We can replace $\sum_{i=1}^n \lambda_i = 1$ by $\max_{1 \leq i \leq n} \lambda_i = 1$ in Proposition 4.5 and 4.6.

By the proof of 2.10 [12], we may assume that an affine definable C^rG-manifold is a definable C^rG-submanifold closed in some representation Ω of G. Thus similar proofs of Lemma 2.14, 2.15 and Theorem 2.16 prove the following.

Theorem 4.7. *If X is a compact affine definable C^rG-manifold and $1 \leq r < \infty$, then every definable C^rG-fiber bundle $\eta = (E, p, X \times [0,1], F, K)$ is definably C^rG-isomorphic to $(p^{-1}(X \times \{0\}) \times [0,1], p', X \times [0,1], F, K)$, where G acts on $[0,1]$ trivially, $X \times \{0\}$ is identified with X and $p' = p|p^{-1}(X \times \{0\}) \times id_{[0,1]}$.*

Theorem 1.5 follows from Theorem 4.7.

The following result is a definable C^rG-version of Theorem 3.4, which is obtained similarly.

Theorem 4.8. *Let $\eta = (E, p, X)$ be a definable C^rG-vector bundle of rank k over an affine definable C^rG-manifold X and $1 \leq r < \infty$. Then the following five properties are equivalent.*

(1) *The bundle η is strongly definable.*
(2) *There exists a surjective definable C^rG-morphism from a trivial G-vector bundle $X \times \Omega$ onto η for some representation Ω of G.*
(3) *There exists an injective definable C^rG-morphism from η to a trivial G-vector bundle $X \times \Omega$ for some representation Ω of G.*
(4) *There exists a definable C^rG-vector bundle η' over X such that $\eta \oplus \eta'$ is definably C^rG-isomorphic to a trivial G-vector bundle.*

(5) *There exist non-equivariant definable C^r-sections $s_1, \dots, s_n : X \to E$ of η such that:*
  (a) *For any $x \in X$, the vectors $s_1(x), \dots, s_n(x)$ generate the fiber $p^{-1}(x)$ over x.*
  (b) *The sections $s_1, \dots, s_n$ generate a finite dimensional G-invariant vector subspace of $\Gamma(\eta)$.*

Proof of Theorem 1.7. Since X is compact, a similar proof of Lemma 2.15 proves that there exist finitely many points $x_1, \dots, x_n \in X$ with definable C^r-slices $S_{x_1}, \dots, S_{x_n}$ and α-dimensional representations $\Omega_{x_1}, \dots, \Omega_{x_n}$ of $G_{x_1}, \dots, G_{x_n}$, respectively, such that $\{GS_{x_i}\}_{i=1}^n$ is a finite definable open G-covering of X and each $\eta|GS_{x_i}$ is definably C^rG-equivalent to $\epsilon(S_{x_i})$, where $\epsilon(S_{x_i}) = (G \times_{G_{x_i}} (S_{x_i} \times \Omega_{x_i}), p, G \times_{G_{x_i}} S_{x_i}), p : G \times_{G_{x_i}} (S_{x_i} \times \Omega_{x_i}) \to G \times_{G_{x_i}} S_{x_i}, p([g, x, y]) = [g, x]$ and α denotes the rank of η. Clearly each $\epsilon(S_{x_i})$ admits finitely many definable C^r-sections satisfying Condition (5) in Theorem 4.8. Thus every $\eta|GS_{x_i}$ admits definable C^r-sections $s_{i1}, \dots s_{it_i}$ satisfying the same condition.

By Proposition 4.6, we have an equivariant definable C^r-partition of unity $\{\lambda_i\}_{i=1}^n$ subordinate to $\{GS_{x_i}\}_{i=1}^n$. Let $\overline{s_{iq}} := \lambda_i s_{iq}$. Then for any $g \in G$, $g \cdot \overline{s_{iq}} = \lambda_i(g \cdot s_{iq})$. Therefore a finite family of definable C^r-sections $\overline{s_{11}}, \dots, \overline{s_{1t_1}}, \dots, \overline{s_{n1}}, \dots, \overline{s_{nt_n}}$ satisfies the required conditions.

Now we prove the second part of the theorem. If η is strongly definable, then there exist a representation Ω of G and a definable C^rG-map f from X to $G(\Omega, \alpha)$ such that η is definably C^rG-isomorphic to $f^*(\gamma(\Omega, \alpha))$. Since the total space of $f^*(\gamma(\Omega, \alpha))$ is affine, E is affine.

Conversely, we assume that E is a definable C^rG-submanifold of a representation Ξ of G.

Let

$$F_1 : X \to M(\Xi), F_1(x) = \text{ the matrix projecting } T_x\Xi \text{ onto } T_xE,$$

$$F_2 : X \to M(\Xi), F_2(x) = \text{ the matrix projecting } T_x\Xi \text{ onto } T_xX.$$

Then by a way similar to the proof of I.3.3 [19], F_1 and F_2 are definable maps. Thus they are definable C^r-maps. By the definition of G-action, they are G-maps. Hence they are definable C^rG-maps. Let

$$F : X \to G(\Xi, \alpha), F = (id - F_2)F_1.$$

Then F is a definable C^rG-map and η is definably C^rG-isomorphic to $F^*(\gamma(\Xi, \alpha))$. Therefore η is strongly definable. □

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Department of Mathematics, Faculty of Education, Wakayama University, Sakaedani Wakayama 640-8510, Japan

E-mail address: kawa@center.wakayama-u.ac.jp